【Article】

The Effect of Self-Gravity in Linearly Perturbed Euler Equations for a Rotating Thin Fluid Disk

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Abstract : The stability of a rotating thin fluid disk bound by weak self-as well as a central gravity is studied within the linearly perturbed Euler equation. The algebraic equation that the eigenfrequencies (EFs) must satisfy is derived and is solved. In absence of the self-gravity within the disk, it has been known that, in addition to the exponential instability caused by the imaginary part of EF, the polynomial instability (PI) exists in which the amplitude of the density perturbation grows linearly in time. The exponential instability is occasional, while the PI is ubiquitous over the disk. The self gravitation of the disk shifts the phase of the density wave and gives rise to a new sinusoidal variation of the EF to the radial direction. The WKB-like approximation shows that the net effect of the self-gravity of perturbations is to increase the central mass.

Keywords : galaxy rotation ; self gravity ; eigenfrequency ; polynomial instability

1. Introduction

The endeavour to understand the hydrodynamical origin of the structure of spiral galaxies started from the works by Lindblad (1948 ; 1964), Toomre (1964), Lin and Shu (1964), and Goldreich and Lynden-Bell (1965). Through successive studies, a close consensus that the arms are density waves seems to have been reached, although such problems as on the age, winding direction, metamorphosis, etc. of the arms still remain unsolved (For reviews, see Binney and Tremaine 2008 ; Sellwood 2014) .

In theoretical studies, spiral galaxies are frequently treated as thin fluid disks. Equipped with the WKB approximation method, the fluid disk models have been widely employed in astrophysical problems because of their tractability (Griv et al. 2008 ; Binney and Tremaine 2008 ; Roshan and Abbassi 2015). There, all radial variations of physical quantities except for the phase oscillation in the radial direction are ignored. Supplemented by the assumption of barotropicity of the fluid, the WKB method provides the dispersion relation and the stability condition that govern the dynamics and the fate of the spiral structure.

One problem in the WKB approximation lies in that, despite of the starting assumption, the resultant dispersion relation is strongly dependent on the radial coordinate. Another is the appearance of singularities in the equations of motion, the Lindblad and the corotation resonances, that require an additional careful prescription to deal with them (Goldreich and Tremaine 1979 ; Lubow and Ogilvie 1998).

In addition to the problem inherent in the WKB method, the density perturbations give rise to another obstacle that renders the study to unveil the nature of solitary galaxies complicated. Namely, the modulations of gravitational interaction among density perturbations alter the pattern of the perturbations themselves.

How does the density wave form and behave? The purpose of this paper is to find an answer to this question by exploring a solvable disk model of spiral galaxies analysed recently by Takahashi (2015). In this model, together with the viscosity expansion method for the compressible fluid, Takahashi found novel vortical solutions with new radial velocity profiles. The azimuthal velocity profile of these solutions survives in the inviscid limit and thus can be a basic state of the corresponding Euler equation. Furthermore, the resultant density profile is such that the integrated mass is proportional to the radial distance from the origin at long distances. The rotation curves of the disk were found to be consequentially consistent with observations.

The stability of the disk model introduced by Takahashi was also explored within the linear perturbations. It is noticeable that the secular equations can be treated rigorously and the eigenfrequencies (EFs) are exactly determined *provided that the self-gravitational interaction can be ignored*. The remarkable fact is that the EFs are generally dependent on r, the distance from the centre of the disk, and is decreasing with r at large distances. The tight winding of arms of the density wave takes place owing to the temporally growing phase $-\omega(r)t$, where t is the time variable. This gives the basis for the WKB approximation. In Takahashi (2015), the results of perturbation analysis are concisely summarized.

The gravity plays the fundamental role in the dynamics of astrophysical disk. In Takahashi (2015), only centripetal gravity exerted by the central mass is considered. Of course, the effects of the self-interaction among the density perturbations are of the greatest interest in the dynamics of galaxy and are remained to be clarified.

In this paper, more general arguments of deriving the local EFs are presented. It will be shown that there exist a large number of solutions with distinctive temporal behaviours, of which the solution found in Takahashi (2015) is of the simplest one. The gravitational interaction among the perturbations will also be taken into consideration through the modulations of the gravitational potential.

This paper is organized as follows. In sec.2, the perturbed Euler equation with the axisymmetric gravity is solved. A new parameter, s, will be found that distinguishes the temporal behaviours of the perturbations. The local EFs are obtained for the simplest case of $s = 0$. A review of the case of the central point mass dominance is presented in sec.3. In sec.4, the algebraic equation for the EF under the non-axisymmetric self-gravity interaction is found. Summary is presented in sec.5.

2. Linear perturbation of the Euler equation

Following Takahashi (2015), we consider a thin fluid disk of an inviscid and circular flow with the inner radius r_1. The velocity field is given in the cylindrical coordinate system (r, θ, z) by

$$\boldsymbol{v} = (0, v_\theta, 0), \tag{2.1}$$

where v_θ is the azimuthal component. The radial and axial components vanish. We examine the linear perturbation of the Euler equation on the background velocity (2.1) under the action of axisymmetric gravity.

The small variations of the velocity field, the pressure and the density denoted by $\delta\boldsymbol{v} = (\delta v_r, \delta v_\theta, \delta v_z)$, δp and $\delta\rho$, respectively, are generated at time $t = 0$. These are functions of radial coordinate r, azimuthal angle θ and t. In this section, for the sake of transparent arguments, we further make three assumptions. First, the gravitational force

$$\boldsymbol{f}_G = -2\pi G \frac{\hat{\boldsymbol{r}}}{r^2} \int_{r_1}^{r} \rho(r') r' dr', \tag{2.2}$$

where $\rho(r)$ is the two-dimensional mass density and G the gravitational constant, that acts on the fluid element at $\boldsymbol{r}$ is dominated by the central force exerted by the axially symmetric mass distribution, so that the self-gravitational force among fluid elements due to a small perturbation can be neglected, i.e., $\delta f_{Gr} = \delta f_{G\theta} = 0$. Second, deformations are restricted to take place within the disk plane and the disk itself does not deform. Third, v_z is fixed to zero on the disk, i.e., $v_z = \delta v_z = 0$. The effect of the self-gravitation is intriguing and will be treated separately in sec.6. Then the temporal evolutions of the perturbations are governed by the linearly perturbed Euler equations

$$\left(\partial_t + \frac{v_\theta}{r}\partial_\theta\right)\delta v_r - \frac{2 v_\theta \delta v_\theta}{r} - \frac{\delta\rho}{\rho}\left(\frac{v_\theta^2}{r} + f_{Gr}\right) + \frac{\partial_r \delta p}{\rho} = 0, \tag{2.3a}$$

$$\left(\partial_t + \frac{v_\theta}{r}\partial_\theta\right)\delta v_\theta + \frac{1}{r}\partial_r(r v_\theta)\delta v_r + \frac{\partial_\theta \delta p}{r\rho} = 0, \tag{2.3b}$$

together with the mass conservation equation

$$\left(\partial_t + \frac{v_\theta}{r}\partial_\theta\right)\delta\rho + \frac{1}{r}\partial_r(r\rho\delta v_r) + \frac{1}{r}\rho\partial_\theta\delta v_\theta = 0. \tag{2.3c}$$

Substituting to (2.3) the density wave form

$$\delta v_r = Ae^{im\theta - i\omega t}, \delta v_\theta = Be^{im\theta - i\omega t}, \delta p = Ce^{im\theta - i\omega t}, \delta\rho = De^{im\theta - i\omega t}, \tag{2.4}$$

for the perturbations, equations (2.3a)~(2.3c) are rewritten as

$$(\partial_t - i\tilde{\omega})A - 2\Omega B + \frac{1}{\rho}(C' - i\omega' tC) - \frac{1}{\rho}(r\Omega^2 + f_{Gr})D = 0, \tag{2.5a}$$

$$(\partial_t - i\tilde{\omega})B + \frac{(r^2\Omega)'}{r}A + \frac{im}{r\rho}C = 0, \tag{2.5b}$$

$$(\partial_t - i\tilde{\omega})D + \rho A' - \left(i\omega' t\rho - \frac{(r\rho)'}{r}\right)A + \frac{im}{r}\rho B = 0, \tag{2.5c}$$

where $\tilde{\omega} \equiv \omega - m\Omega$ and the prime stands for a differentiation in r and $\Omega \equiv v_\theta/r$. The EF ω has an r dependence for the fluid in differential rotation. The differentiation of the Fourier factor with respect to r therefore yields the factor $i\omega' t$. Note that, if the set of (2.5a) and (2.5b) is solved for static amplitudes A and B in terms of C and D, then the solution for the amplitudes has the factor $\Delta = 1/(\kappa^2 - \tilde{\omega}^2)$ with $\kappa \equiv 2\sqrt{|\Omega(\Omega + r\Omega'/2)|}$ being the epicycle frequency, thereby giving rise to the well-known Lindblad resonances at $\kappa^2 = \tilde{\omega}^2$. Any divergence should not be involved in reality. Since there exist explicit t-dependences in (2.5a) and (2.5c), some of the four amplitudes will have temporal dependences. It is such t-dependences which render our system of equations free from divergences mentioned above.

Before entering into details of our arguments, it may be instructive to recall the case of incompressible vortex studied by Kelvin (1880) for $v_\theta \propto r$ and Synge (1933) for more general r-dependence (See also Ash and Khorrami 1995; Takahashi 2013). They considered the three-dimensional perturbations on the background field (2.1) of incompressible fluid and the corresponding pressure. The perturbations were expressed in terms of a normal Fourier mode with respect to θ, z and t. Then, the flow turns out to be stable for axisymmetric perturbations when $v_\theta(rv_\theta)'$ is positive at any spatial point. In such a situation, ω is real and constant and the explicit t-dependences in (2.5a) and (2.5c) disappear. For nonaxisymmetric perturbations, the stability condition has been found to be more intricate and does not seem to be suited to practical use (Ash and Khorrami 1995).

In the followings, by preserving the terms with explicit t-dependence in (2.5a) and (2.5c), thereby focusing our interest to the case that $\omega(r)$ is a nontrivial function of r, we will find new solutions adapted to compressible fluid. The result for the simplest case has been summarized in Takahashi (2015). Below, we want to generalize the arguments.

The terms with explicit t-dependence in (2.5a) and (2.5c) must be cancelled by some t-dependences in the amplitudes. We may try the simplest polynomial forms

$$A = \sum_{i=s-3}^{s} A_i t^i, \tag{2.6a}$$

$$B = \sum_{i=s-3}^{s} B_i t^i, \tag{2.6b}$$

$$C = \rho \sum_{i=s-3}^{s} C_i t^i, \tag{2.6c}$$

$$D = \rho \sum_{i=s-2}^{s+1} D_i t^i, \tag{2.6d}$$

where the powers of t are positive and the coefficients A_i etc. are functions of r only. The temporal behaviour of the density that is polynomial in t was already noticed by Goldreich and Lynden-Bell (1965), but for ω= 0 only (For polynomial instability of plasma, see Smith and Rosenbluth 1990).

In (2.6), each of the amplitudes A, B, C and D involves four unknown coefficients so that totally sixteen unknown coefficients are present, while substituting (2.6) to (2.5) gives sixteen equations. Therefore, the polynomial expressions (2.6) for the amplitudes are generally necessary and sufficient to find non-trivial solutions. See Appendix A for details. In the followings, we restrict ourselves to the minimal case of s = 0 with the non-vanishing coefficients being A_0, B_0, C_0, D_0 and D_1. Specifically, D is a linear function of t :

$$D = D_0 + D_1 t\ . \tag{2.7}$$

For s = 0, there exist five equations that are linear in five unknown functions A_0, B_0, C_0, D_0 and D_1, in which the first order derivatives of A_0 and C_0 are involved. When the unperturbed density is uniform and $\rho' = 0$, it is possible to eliminate those derivatives, thereby reducing (2.5) to simultaneous linear algebraic equations of unknown functions that do not involve their derivatives. Then, by requiring the existence of nontrivial solutions, it is straightforward to derive the algebraic eigenvalue equation for ω :

$$Z^4 - \left(2 + 2a - \frac{(rf_{Gr})'}{r\Omega^2}\right) Z^2 - 4maZ - (ma)^2 = 0, \tag{2.8a}$$

where

$$Z \equiv \frac{\omega(r)}{\Omega(r)} - m, \tag{2.8b}$$

$$a \equiv 1 + \frac{f_{Gr}(r)}{r\Omega(r)^2}, \quad a \leq 1. \tag{2.8c}$$

Here $\Omega Z/m$ is the Doppler-shifted angular frequency and f_{Gr} is the r-component of the gravitational force (2.2). A concise explanation on the derivation of (2.8) has been given in Takahashi (2015) and is elaborated in Appendix A.

For a given integer m, there generally exist four solutions, two of which are always real. Both of the other two are either real or complex. The EFs are obtained from Z as

$$\omega = \omega(r) = (m + Z(r))\Omega(r). \tag{2.9}$$

The density wave propagates to the azimuthal direction for $m > 0$. Its angular velocity is given by

$$\Omega_{\mathrm{p}} = \frac{\omega}{m} = \left(1 + \frac{Z}{m}\right)\Omega. \tag{2.10}$$

The relative speed of the density wave to the fluid flow is given by $Z\Omega r/m$.

$-\mathrm{Re}\omega t$ is the so-called shape function by which the spiral pattern of the perturbation is determined (Binney and Tremaine 2008). In general, $\mathrm{Re}\omega$'s are functions of r that vanishes at long distances. As t gets large, therefore, the phase factor $\exp(-i\mathrm{Re}\omega t)$ rapidly oscillates with the change of r over the disk. This leads to the tight winding of arms, the situation for the WKB approximation to be validated (Binney and Tremaine 2008).

In the WKB approximation, the perturbations are expanded in Fourier components as

$$a(r)\exp[ikr - i\omega(k)t] \tag{2.11}$$

where the amplitude a is assumed to be a slowly varying function of r, while, with $\mathrm{Re}k$ being large, the phase factor $\exp(ikr)$ is rapidly oscillating. $\omega(k)$ in (2.11), assumed to have no coordinate dependence, gives the dispersion relation. We have seen that the correct phase factor is $\exp[-i\omega(r)t]$ where $\omega(r)$ given by (2.9) has a non-trivial r-dependence. Does this phase factor have a Fourier decomposition like (2.11)? The answer is generally *no* except for special cases. This problem is elaborated in Appendix C.

The stability of the axisymmetric perturbation is manifested by setting $m = 0$ in (2.8a). The EF of the unstable mode is given by $Z^2 = 0, 2 + 2a - (rf_{Gr})'/(r\Omega^2)$. The former gives the static solution $\omega = 0$, which exhibits no temporal dependence (See Appendix B). Concerning the latter, we have

$$\omega^2 = 4\Omega^2 + \frac{1}{r}[f_{Gr} - (rf_{Gr})']. \tag{2.12}$$

If we choose the Newtonian central gravity (2.2) for f_{Gr}, (2.12) reads

$$\omega^2 = 4\Omega^2 - \frac{2\pi G\rho}{r}\left(\frac{M_r}{\pi r^2 \rho} - 1\right), \tag{2.13}$$

where $M_r = 2\pi\int_0^r \rho(r')r'dr'$ is the mass inside the radius r. (2.13) may be compared with the WKB result (Binney and Tremaine 2008)

$$\omega_{\mathrm{WKB}}^2 = 4\Omega^2 + 2r\Omega\Omega' - 2\pi G\rho|k| + v_s{}^2k^2, \tag{2.14}$$

where k is the radial wave number and, with p being the pressure, $v_s = (dp/d\rho)^{1/2}$ is the sound velocity.

The sum of the first two terms is the square of the epicycle frequency. (2.14) with $\omega_{\mathrm{WKB}}^2 < 0$ gives the Toomre's instability condition (Toomre 1964). The appearance of k is due to replacing the derivative d/dr in the perturbation equations by ik.

The wave number k does not appear in our model of $s = 0$. This is because, when $s = 0$, the proper equation for the EF is not affected from the derivatives of amplitudes. The situation is intricate in the case of $s \neq 0$, in which it is impossible to delete the derivative of amplitudes from the linearized perturbation equations. Unfortunately, it is quite difficult to determine how ω depends on s, which is an obstacle in finding the general stability criterion in our model.

The relation between the pressure and the density has been determined not a priori but by the Euler equation. This seems reasonable for weak gravity because, in that case, the density can vary kinematically without affecting the pressure. Thus the sound velocity v_s that is usually determined from the barotropic equation of state does not appear in (2.13).

The system of equations (2.5a)~(2.5c) and (2.6) are sufficient to obtain the exact and tractable expression for the solution. It is also applicable at the resonances. In fact, we can easily find finite solutions at the Lindblad and corotation resonance. See Appendix D for details.

3. Solutions to (2.8) and the polynomial instability

The eigenfrequency equation (2.8) has been solved in Takahashi (2015) for some cases of gravity strength. It may be interesting that complex modes emerge for $|m| \geq 2$ when the gravity due to the central mass is weak and the disk mass is totally ignored.

Below, we summarize the results for finite gravity.

3.1 Central mass dominance

Let us consider the case in which a mass M_c locates at the centre of the disk of low density and is exerting the gravitational force $f_{Gr} = -GM_c/r^2$ to the fluid. The interactions among disk masses are neglected. Noting that $(rf_{Gr})' = -f_{Gr}$ for the Newtonian gravitational force, we parameterize a given by (2.8c) as

$$a = 1 + \frac{f_{Gr}}{r\Omega^2} \equiv 1 - \frac{\alpha}{r^3\Omega^2}, \tag{3.1}$$

and rewrite (2.7a) as

$$Z^4 - (1+3a)Z^2 - 4maZ - (ma)^2 = 0. \tag{3.2}$$

For a given $\Omega(r)$, it is easy to find numerically the EFs from the roots of (3.2) for arbitrary a defined

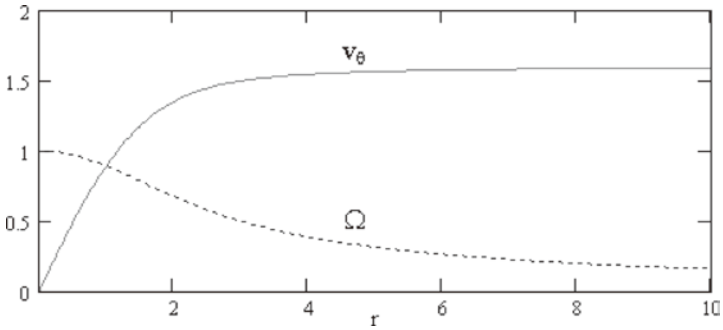

Fig. 1　The model rotation curve v_θ (solid curve) and the corresponding circular frequency Ω (dotted curve) used for calculations of EFs.

by (3.1).　The results have been given by Takahashi (2015).　In the followings, we argue the case of the flat rotation curve presented in Fig. 2 in the above reference.

The units chosen are 10^4 ly$=3.1\times10^3$ pc and 100 km/s for the length and the velocity, respectively. The unit of time is therefore 10^4 ly/100 km/s$=3\times10^7$ yr.　With these units, writing the central mass as N times the solar mass, α is given by

$$\alpha = 1.3\times10^{-10}N. \tag{3.3}$$

Some typical examples of ω for α=10 and $m = 2$ are depicted in Fig. 2 together with $\omega_{L\pm} = m\Omega\pm\kappa$ and $\omega_C = m\Omega$ that correspond to the Lindblad and the corotation resonances, respectively.　The singularities at $\omega = \omega_{L\pm}, \omega_C$,which we shall call the $L_\pm$- and C-singularity, respectively, are inherent in the self-consistent WKB approximation (See, e.g., Binney and Tremaine 2008). From Fig. 2(a), we see that it is impossible to avoid all of the singularities.　There exist a bifurcation point A and a coalesce point B.　The gross basic patterns of ω's do not change for other choice of parameter values (except for $m = 0$).

Six branches of $\mathrm{Re}\omega^{(j)}$, j = I, II, III, IV, V, VI, are observed in Fig. 2(a).　The branches I, III, IV and V are real and are of neutral stability.　The branches II and VI are of complex ones, each of which are denoted as II, II', VI and VI', where the prime denotes the branch with a negative imaginary part.　$\omega^{(I)}$ is real and decreases monotonically as r increases.　$\omega^{(II)}$ bifurcates at A ($r = r_A \approx 3.8$) to real $\omega^{(IV)}$ and real $\omega^{(V)}$.　$\omega^{(III)}$ and $\omega^{(IV)}$ coalesce to $\omega^{(VI)}$ at B ($r = r_B \approx 4.2$).　The complex ω's are of exponentially growing and decaying modes for δv_r, δv_θ and δp.

The density perturbation evolves in a way different from the velocity and pressure.　Because the amplitude D given by (2.7) is a linear function of time, there exists a section AB in which the eigenfrequencies are all real but D of all modes grow linearly in time.　This is the 'polynomialinstability

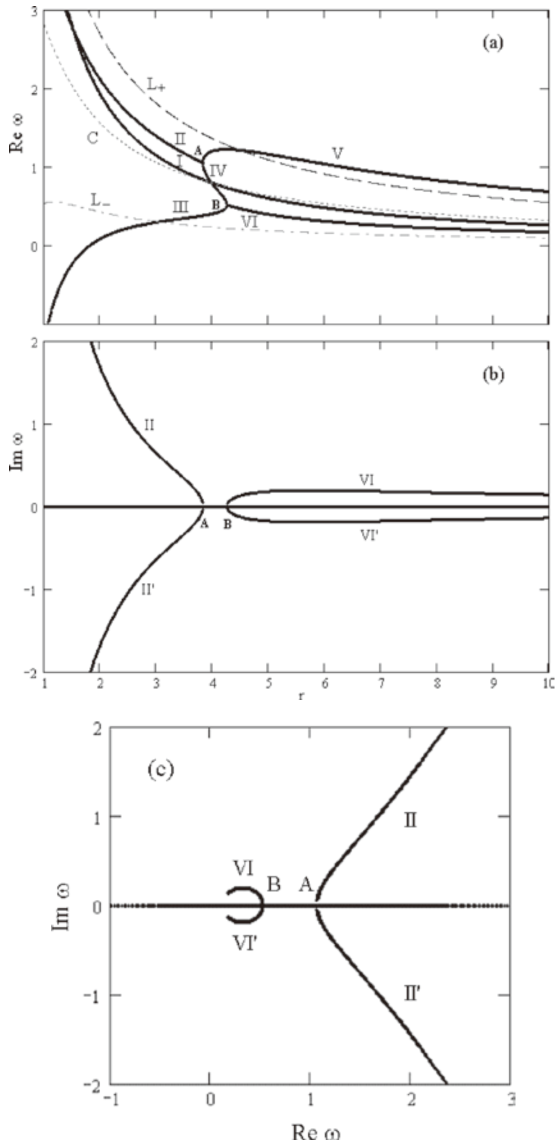

Fig. 2 Solutions of (5.10) for the rotation curve with $k_1/k = 1$ in Fig. 2 and $m = 2$. (a) Reω vs. r. $\omega_{L\pm}$ (dashed and dot-dashed curves) and ω_C (dotted curve) are labelled by $L_\pm$ and C, respectively. (b) Imω vs. r. (c) Imω vs. Reω. Parameter values are $\alpha = 10$, $k = 10$ ly^{-2}. v_θ/r is also shown by dotted curve in (a).

section (PIS)' (Takahashi 2015). This linear growth is attributable to the modulation of matter velocity, by which fast moving matter catches up with or runs ahead of slow matter. Concerning the density, all of the four modes are 'unstable' in short term because of such one-way amplifications of oscillations.

Obviously, the growth rate of perturbations is smaller in PIS than inner or outer region of exponen-

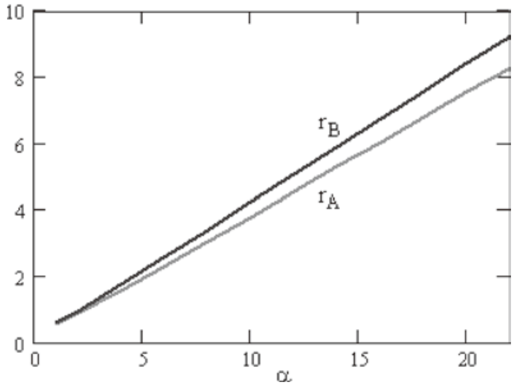

Fig. 3　r_A and r_B as functions of α.　Values of other parameters than α are same as those in Fig. 4.

tial growth.　In reality, no discrepancy in the growth rates of spiral arms in a single spiral galaxy is observed.　In other words, the PIS seems not to be preferred in real galaxies.

The bifurcation point A and the coalesce point B shift left or right as α gets smaller or larger, respectively.　We observe approximate linear dependences of r_A and r_B on α, particularly for large α, as is shown in Fig.3. $\alpha = 20$, which corresponds to 15×10^{10} solar mass, gives $r_A > 7 \times 10^4$ ly.　Sufficiently large central mass expels the PIS too far from the centre of the disk to be observed.

3.2　Finite disk mass

In case α is small, the mass of the disk will play a role.　This corresponds to the existence of the dark matter.　Here, we aim at getting an insight into the effect of the disk mass by referring to (2.2) for the expression of gravitational force.　After small variations were taken, the strongest inter-matter interaction will be the one between $\delta\rho(r)$ and the mass inside the radius r

$$M_d(r) = 2\pi \int_{r_i}^{r} \rho(r') r' dr'. \tag{3.4}$$

Since M_d is a function of r only, we may express an approximate gravitational force in the variational equation (2.3a), i.e.,

$$f_{Gr} = -\frac{\alpha}{r^2} - \frac{GM_d(r)}{r^2}. \tag{3.5}$$

Concerning $M_d(r)$, several kinds of behaviour are possible near the disk centre, while, for $r \to \infty$, M_d grows linearly in r (Takahashi 2015).　Once $\rho(r)$ is fixed, finding the EFs numerically is a rather easy task. Grossly speaking, the distributions of EFs are determined by $\alpha' = \alpha + < GM_d >$, where $< GM_d >$ is some characteristic value of $GM_d(r)$.　In other words, the stability of the massive disk is very similar to the one that consists of a central mass with a modification $\alpha \to \alpha'$ that has been considered in the previous subsection.

4. Effect of perturbed gravitation

We here estimate the effects of the self-gravitation due to the density variation within the disk. The perturbation equations (2.3a) and (2.3b) are modified by equating l.h.s. of these equations to the gravity modulation δf_{Gr} and $\delta f_{G\theta}$, respectively, where $(\delta f_{Gr}, \delta f_{G\theta}) = -(\partial_r, r^{-1}\partial_\theta)\delta\Phi$ and

$$\delta\Phi(\boldsymbol{r}, t) = -\frac{G}{4\pi}\int \frac{\delta\rho(\boldsymbol{r})}{|\boldsymbol{r}'-\boldsymbol{r}|}d\boldsymbol{r}', \tag{4.1}$$

where $\delta\rho(\boldsymbol{r}) = \delta\rho(r)\delta(z)$ with $\delta\rho(r)$ being given by (2.4).

The resultant integro-differential equations may be converted to the higher order differential equations with use of Poisson's equation $\nabla^2\delta\Phi = 4\pi G\delta\rho(\boldsymbol{r})$. The eigenvalue equations corresponding to (2.5) now involve terms of the second and third powers of t, which renders the problem awkward. Instead, in order to see the effect of self-gravitation, it is convenient to seek an approximate expression for $\nabla\delta\Phi$ that has an explicit factor $e^{im\theta - i\omega_{mt}}$ that is common to all the perturbation amplitudes. For this purpose, with the help of the Gauss integration formula, we express (4.1) as

$$\begin{aligned}\delta\Phi(\boldsymbol{r}, t) &= -\frac{G}{4\pi}\int \frac{D(r', t)e^{im\theta' - i\omega(r')t}}{\sqrt{r'^2 - 2rr'\cos(\theta'-\theta)+r^2}}r'dr'd\theta' \\ &= -\frac{Ge^{im\theta}}{\sqrt{\pi}}\int_0^\infty\int_0^\infty D(r', t)e^{-i\omega(r')t}e^{-(r'^2+r^2)s^2}I_m(2rr's^2)r'dr'ds\end{aligned} \tag{4.2}$$

where I_m is the modifies Bessel function of the first kind.

We restrict ourselves to the case that there exist the wound spiral arms of density modulation, i.e., $m \geq 1$ and to the point far from the core region, i.e., $r > r_1$ where r_1 is the inner radius of the disk. Furthermore, we assume that $r'D(r')I_m(2rr's^2)$ rapidly approaches zero as $r' \to 0$ and the integration in (4.2) is governed by the form $I_m(z) \sim e^z/\sqrt{2\pi z}$ with a restriction $|z| > 1$. In terms of the variables r' and s, this implies $s > 1/\sqrt{2rr'}$. Then $\delta\Phi$ may be approximated by

$$\delta\Phi \sim -\frac{Ge^{im\theta}}{2\pi\sqrt{r}}\int_{r_1}^\infty \sqrt{r'}\,dr'\int_{1/\sqrt{2rr'}}^\infty D(r', t)e^{-i\omega(r')t}e^{-(r'-r)^2s^2}\frac{ds}{s}. \tag{4.3}$$

Next, the r'-integration is performed by replacing $e^{-(r'-r)^2s^2}$ by $\delta(r'-r)\sqrt{\pi}/s$, which will be valid when $|D(r)|$ is very small near r_1 and $\omega(r)$ is slowly varying. Thus we have

$$\delta\Phi \sim -\frac{Ge^{im\theta}}{2\pi}\int_{1/(\sqrt{2}r)}^{\infty} D(r,t)e^{-i\omega(r)t}\frac{\sqrt{\pi}}{s}\frac{ds}{s}$$
$$= -\frac{GD}{\sqrt{2\pi}}re^{im\theta-i\omega t}. \tag{4.4}$$

If the factor r on the r.h.s. of (4.4) is replaced by ΔR, the pitch of winding of spiral arms, then the expression (4.4) is essentially same as the expression obtained by the WKB approximation (see, e.g., Binney and Tremaine 2008). Write $\omega_1 = \mathrm{Re}\omega$. In the decreasing region of ω_1, then, ΔR for trailing spirals in our model is related to ω by

$$\omega_1(r+\Delta R)-\omega_1(r) = -2\pi m/t, \quad |m| \geq 1, \tag{4.5}$$

Expanding $\omega_1(r+\Delta R)$ in ΔR up to the second order, we have

$$\Delta R \approx -\frac{1}{\omega_1''}\left(\omega_1' + \sqrt{\omega_1'^2 - 4\pi m\omega_1''/t}\right). \tag{4.6}$$

For $\omega_1 \approx w_1/r^p$ with $w_1 > 0$ and $p>0$, (4.6) leads to

$$\Delta R \approx \frac{r}{p+1} - r\sqrt{\frac{1}{(p+1)^2} - \frac{4\pi mr^p}{w_1 p(p+1)t}}. \tag{4.7}$$

In the region where $r^p/t \sim w_1 p/(4\pi m(p+1))$, therefore, (4.4) is equivalent to the result by the WKB approximation.

The contribution of perturbation to each component of $\boldsymbol{f}_G$ is given by

$$\delta f_{Gr} = -\partial_r\delta\Phi \sim \frac{r}{\sqrt{2\pi}}\left(\frac{D'}{D} - i\omega_m' t\right)GDe^{im\theta-i\omega t}, \tag{4.8a}$$

$$\delta f_{G\theta} = -\frac{1}{r}\partial_\theta\delta\Phi \sim \frac{im}{\sqrt{2\pi}}GDe^{im\theta-i\omega t}. \tag{4.8b}$$

Placing (4.8a) and (4.8b) with their Fourier factors being deleted on the r.h.s. of (2.5a) and (2.5b), respectively, yields the modified equations of perturbations. In this modification, a replacement of gravitational force

$$f_{Gr} \to f_{Gr} + \frac{r}{\sqrt{2\pi}}\left(\frac{D'}{D} - i\omega_m' t\right)G\rho \tag{4.9}$$

is taking place in (2.5a). Since f_{Gr} is negative, this replacement acts to enhance (suppress) the effect of gravity by the central mass if D'/D is positive (negative) as long as the radial motion is concerned. As the time elapses, the change in the phase indefinitely develops.

A, iB, iC and D are chosen to be real when $\delta\boldsymbol{f}_G$ is neglected. Therefore (2.5) and (4.8) mean that, concerning the azimuthal motion, the self-gravitation gives rise to the phase shifts. An additional effect of $\delta f_{G\theta}$ on (2.5) is that the amplitude A, B or C will acquire a term linear in t because D with $m \geq 1$ is already linear in t.

The derivation of the eigenfrequency equation is straightforward but cumbersome. The details are relegated to Appendix E. The result is

$$Z^4-\left(\frac{(\Pi+\varepsilon)'}{\Omega^2}+\frac{\Pi+\varepsilon}{r\Omega^2}+\frac{2r\Omega'}{\Omega}+4\right)Z^2+4m\frac{\Pi+\varepsilon}{r\Omega^2}Z-m^2\left(\frac{\Pi+\varepsilon}{r\Omega^2}\right)^2=0, \qquad (4.10)$$

where $\Pi\equiv -r\Omega^2-f_{Gr}=-r\Omega^2 a$ and $\varepsilon\equiv\left(Gr\rho/\sqrt{2\pi}\right)'$. Z and a have been defined by (2.8b) and (2.8c), respectively. (4.10) reduces to (2.8a) for $\varepsilon=0$.

It has been proved in Takahashi (2015) that $\rho \propto 1/r$ for large r, so that ε will asymptotically behave as $1/r^2$. Furthermore, on the basis of the analyses in Takahashi (2015), it is easy to show that the next-to-leading term is d_{-2}/r^2 with $d_{-2}>0$ for the family of solutions with constant rotation curves. Therefore, in such solutions, $(r\rho)'\propto 1/r^2<0$ for $r\to\infty$, meaning that the effect of the intra-disk interactions at long distances on the eigenfrequencies is equivalent to increasing the central mass of the disk. The increase of the central mass is reflected as the increase of the parameter α in (3.1). From Fig. 3, this implies that the PIS is expelled toward the outer region of the disk.

5. Summary

The linearly perturbed Euler equations were solved for an axially symmetric and differentially rotating thin fluid disk bound by a central gravity and/or pressure gradient. The self-gravitational interaction among the fluid elements was approximately taken into account. The algebraic equation for the EFs was obtained. The EFs depend on a parameter s which specifies the polynomial structure of amplitudes in time variable t. The case of $s=0$ that corresponds to the previous work was solved. The EFs are all real in some cases, can be complex in the other, depending on the relation among the gravitational force, centrifugal force and pressure gradient. In all cases, the amplitude of density perturbation grows linearly in time, irrespective of the form of rotation curve. In this sense, the axially symmetric rotating thin fluid disk is unstable.

The self-gravity due to perturbations is approximately estimated by the saddle point method and is found to yield an effect equivalent to increasing the central mass. Accordingly, the PIS is shifted further outward of the disk.

The system considered does not have one to one correspondence between the density and the pressure. Namely, the fluid must be baroclinic. Specifically, the density modulation takes place as a consequence of velocity modulation and occurs at zero pressure. This is the reason that the stability condition of our system does not take the form of Toomre's (Toomre 1964).

Appendix A : Equations for the amplitude coefficients in (2.6)

In this appendix, we derive the equations for the coefficients in the polynomials (2.6). Substituting (2.6) to (2.5) and matching the coefficients of the same powers of t, we have (All suffices are zero or positive integer.)

$$(i+1)A_{i+1}-i\tilde{\omega}A_i-2\Omega B_i+C'_i+\left(\frac{\rho'}{\rho}-i\omega'\right)C_{i-1}+\Pi D_i=0, \tag{A1}$$

$$(i+1)B_{i+1}-i\tilde{\omega}B_i+\frac{(r^2\Omega)'}{r}A_i+\frac{im}{r}C_i=0, \tag{A2}$$

$$(i+1)D_{i+1}-i\tilde{\omega}D_i+A'_i-i\tilde{\omega}A_{i-1}+\frac{(r\rho)'}{r\rho}A_i+\frac{im}{r}B_i=0, \tag{A3}$$

where $-\Pi\equiv r\Omega^2+f_{Gr}$ is the sum of the centrifugal force and the gravity (per unit mass). In other words, Π is the pressure gradient per unit mass of the unperturbed system. Here the suffix of each amplitude stands for the associated power of t. (A1) ~ (A3) are rewritten as

$$(s-3)A_{s-3}=0, \tag{A4}$$

$$(s-2)A_{s-2}-i\tilde{\omega}A_{s-3}-2\Omega B_{s-3}+C_{s-3}'+\frac{\rho'}{\rho}C_{s-3}=0, \tag{A5}$$

$$(s-1)A_{s-1}-i\tilde{\omega}A_{s-2}-2\Omega B_{s-2}+C_{s-2}'+\frac{\rho'}{\rho}C_{s-2}-i\omega' C_{s-3}+\Pi D_{s-2}=0, \tag{A6}$$

$$sA_s-i\tilde{\omega}A_{s-1}-2\Omega B_{s-1}+C_{s-1}'+\frac{\rho'}{\rho}C_{s-1}-i\omega' C_{s-2}+\Pi D_{s-1}=0, \tag{A7}$$

$$i\tilde{\omega}A_s+2\Omega B_s-C_s'-\frac{\rho'}{\rho}C_s+i\omega' C_{s-1}-\Pi D_s=0, \tag{A8}$$

$$i\omega' C_s-\Pi D_{s+1}=0, \tag{A9}$$

$$(s-3)B_{s-3}=0, \tag{A10}$$

$$(s-2)B_{s-2}-i\tilde{\omega}B_{s-3}+\frac{(r^2\Omega)'}{r}A_{s-3}+\frac{im}{r}C_{s-3}=0, \tag{A11}$$

$$(s-1)B_{s-1}-i\tilde{\omega}B_{s-2}+\frac{(r^2\Omega)'}{r}A_{s-2}+\frac{im}{r}C_{s-2}=0, \tag{A12}$$

$$sB_s-i\tilde{\omega}B_{s-1}+\frac{(r^2\Omega)'}{r}A_{s-1}+\frac{im}{r}C_{s-1}=0, \tag{A13}$$

$$i\tilde{\omega}B_s-\frac{(r^2\Omega)'}{r}A_s-\frac{im}{r}C_s=0, \tag{A14}$$

$$(s-2)D_{s-2}+A_{s-3}'+\frac{(r\rho)'}{r\rho}A_{s-3}+\frac{im}{r}B_{s-3}=0, \tag{A15}$$

$$(s-1)D_{s-1}-i\tilde{\omega}D_{s-2}+A_{s-2}'-i\omega' A_{s-3}+\frac{(r\rho)'}{r\rho}A_{s-2}+\frac{im}{r}B_{s-2}=0, \tag{A16}$$

$$sD_s - i\tilde{\omega}D_{s-1} + A_{s-1}{}' - i\omega' A_{s-2} + \frac{(r\rho)'}{r\rho}A_{s-1} + \frac{im}{r}B_{s-1} = 0, \tag{A17}$$

$$(s+1)D_{s+1} - i\tilde{\omega}D_s + A_s' - i\omega' A_{s-1} + \frac{(r\rho)'}{r\rho}A_s + \frac{im}{r}B_s = 0, \tag{A18}$$

$$\tilde{\omega}D_{s+1} + \omega' A_s = 0. \tag{A19}$$

From (A4), (A10) and (A15), we have $A_{s-3}=B_{s-3}=D_{s-2}=0$ for general s, thereby leaving thirteen equations for thirteen unknowns.

Since these equations involve the derivatives of amplitudes, further approximation that replaces the derivative by ik is usually invoked. This is noting but the one that consists of the WKB method. However, when $s = 0$, (A4) ~ (A19) enables us to derive the exact algebraic equation that the EFs satisfy as is explained in Appendix B.

Appendix B : The case of $s = 0$

In this appendix, we derive (2.8a) for the eigenfrequency ω. Let us consider the case of $s = 0$ in Appendix A. Recalling that the coefficients with a negative suffix identically vanish, (A4) ~ (A19) reduce to

$$\Pi D_1 = i\omega' C, \tag{B1}$$

$$\tilde{\omega}D_1 + \rho\omega' A = 0, \tag{B2}$$

$$i\tilde{\omega}A + 2\Omega B - \frac{\Pi}{\rho}D_0 = \frac{1}{\rho}C', \tag{B3}$$

$$i\tilde{\omega}B = \frac{(r^2\Omega)'}{r}A + \frac{im}{r\rho}C, \tag{B4}$$

$$i\tilde{\omega}D_0 - D_1 - \frac{(r\rho)'}{r}A - \rho A' - \frac{im}{r}\rho B = 0. \tag{B5}$$

Here $A \equiv A_0$, $B \equiv B_0$ and $C \equiv C_0$. As mentioned above, the wave number k may be introduced in (B3) and (B5) by the replacement $d/dk \to ik$. We will see below that ω for $\sigma=0$ does not depend on k.

There exist five equations for five unknown functions A, B, C, D_0 and D_1. By using (B1) and (B2), C and D_1 are expressed in terms of A, thereby C and D_1 can be eliminated from (B3)~(B5) to yield

$$\left[\frac{\tilde{\omega}}{\rho\Pi}\left(\frac{\rho\Pi}{\tilde{\omega}}\right)' - \frac{\tilde{\omega}^2}{\Pi}\right]A + 2i\Omega\frac{\tilde{\omega}}{\Pi}B = i\tilde{\omega}\frac{D_0}{\rho} - A', \tag{B6}$$

$$i\tilde{\omega}B = \left[\frac{(r^2\Omega)'}{r} - \frac{m}{r}\frac{\Pi}{\tilde{\omega}}\right]A, \tag{B7}$$

$$i\tilde{\omega}\frac{D_0}{\rho}-A'+\left[\frac{\omega'}{\tilde{\omega}}-\frac{(r\rho)'}{r\rho}\right]A-\frac{im}{r}B=0. \tag{B8}$$

From these equations, $i\tilde{\omega}D_0/\rho - A'$ is eliminated and we have two linear equations of A and B :

$$G\begin{pmatrix} A \\ iB \end{pmatrix} \equiv \begin{pmatrix} \frac{\tilde{\omega}}{\rho\Pi}\left(\frac{\rho\Pi}{\tilde{\omega}}\right)'-\frac{\tilde{\omega}^2}{\Pi}+\frac{\omega'}{\tilde{\omega}}-\frac{(r\rho)'}{r\rho} & 2\Omega\frac{\tilde{\omega}}{\Pi}-\frac{m}{r} \\ \frac{(r^2\Omega)'}{r}-\frac{m}{r}\frac{\Pi}{\tilde{\omega}} & -\tilde{\omega} \end{pmatrix}\begin{pmatrix} A \\ iB \end{pmatrix}=0. \tag{B9}$$

Note that no derivatives of amplitudes are involved in (B9), which enables us to resort to no further approximation. Nontrivial solutions exist when det $G = 0$ or

$$\frac{\tilde{\omega}^2}{\rho\Pi}\left(\frac{\rho\Pi}{\tilde{\omega}}\right)'-\frac{\tilde{\omega}^3}{\Pi}+\omega'-\frac{(r\rho)'}{r\rho}\tilde{\omega}+\left(2\Omega\frac{\tilde{\omega}}{\Pi}-\frac{m}{r}\right)\left[\frac{(r^2\Omega)'}{r}-\frac{m}{r}\frac{\Pi}{\tilde{\omega}}\right]=0. \tag{B10}$$

(2.7a) in the text is obtained by rearranging (B10).　D_0, the initial density perturbation, remains undetermined.

In the text, we have seen that the EF equation for $m = 0$ gives a solution $\omega=\tilde{\omega}= 0$.　In this case, however, the matrix elements of G are undetermined, so that we have to go back to the original equations (B1)~(B5).　We readily have

$$A=D_1=0, \tag{B11}$$

$$C'+\Pi D_0=2\rho\Omega B. \tag{B12}$$

(B11) means that the amplitude of the density perturbation does not have the t-dependence. (B12) expresses the balance of the pressure gradient and the centrifugal force.

Appendix C : Impossibility of Fourier transformation of $\exp[-i\omega(r)t]$

Here, we show that the double Fourier transformation of $\exp[-i\omega(r)t]$ generally does not exist. Suppose that $\omega(r)$ is a real, continuous and non-constant monotonic function of r and that the expression

$$e^{-i\omega(r)t}=\iint a(k,\nu)e^{ikr-i\nu t}dkd\nu \tag{C1}$$

is possible. Here the integration in (C1) is supposed to be well-defined, so that the order of integrations is always interchangeable. Multiplying both sides by $\exp(i\nu t)$ and integrating them with respect to t over $-\infty < t < +\infty$, we have

$$\delta(\nu-\omega(r))=\int a(k,\nu)e^{ikr}dk. \tag{C2}$$

Further multiplication by $\exp(-ikr)$ and integration with respect to r over $-\infty < r < +\infty$ yields

$$a(k,\nu)=\frac{1}{2\pi}\int\delta(\nu-\omega(r))e^{-ikr}dr=\frac{1}{2\pi}\frac{1}{|w_1(\nu)|}e^{-ikr_1(\nu)}, \tag{C3}$$

where r_1 is the root of $\omega(r)=\nu$ and $w_1=d\omega/dr/_{r=r_1}$. $r_1(\nu), w_1(\nu)$ and $a(k,\nu)$ are continuous functions of ν. (C3) is the formal expression for the Fourier transform of $\exp(-i\omega(r)t)$.

Now, evaluating (C1) at $t = 0$ yields

$$1=\iint a(k,\nu)e^{ikr}dkd\nu=\iint a(x/r, ry)e^{ix}dxdy, \tag{C4}$$

where the integration variables are changed by $x = kr$ and $y = \nu/r$. The integration in (C4) must not depend on r. This is possible when the dependences of $a(x,y)$ on x and y appear through xy, i.e., $a(k, \nu)$ is a function of $k\nu$. Together with (C3), this requires w_1 be a constant and r_1 be proportional to ν. Since r_1 is a root of $\omega(r)=\nu$, the proportionality of r_1 to ν is assured when ω is a linear function, or

$$\omega(r)=w_1 r. \tag{C5}$$

Of course, constant ω is also allowed, i.e.,

$$a(k,\nu)=\delta(k)\delta(\nu-\omega). \tag{C6}$$

For other forms of ω, consistent Fourier transformation of the form given by (C1) will not exist.

Appendix D : Resonances

There exist two cases in which the system (A1)~(A5) must be treated carefully for nontrivial solutions.

1. Lindblad resonances : Let the external force and the pressure be zero, i.e., $C=\Pi=0$. Then we have

$$\tilde{\omega}D_1+\rho\omega' A=0, \tag{D1}$$

$$\tilde{\omega}A-2\Omega iB=0, \tag{D2}$$

$$\tilde{\omega}iB=\frac{\partial_r(r^2\Omega)}{r}A, \tag{D3}$$

$$A'+\frac{\partial_r(r\rho)}{r\rho}A+\frac{m}{r}iB-\frac{\tilde{\omega}}{\rho}iD_0+\frac{1}{\rho}D_1=0, \tag{D4}$$

where $\tilde{\omega}\equiv\omega-m\Omega$. (D2) and (D3) lead to the condition for the Lindblad resonances, $\tilde{\omega}^2=\kappa^2$, where κ is the epicycle frequency. The amplitudes at the resonance are obtained by solving, e.g., the equation for A :

$$\frac{A'}{A}+\frac{(r\rho)'}{r\rho}+\frac{1}{\tilde{\omega}}\left(\omega'+\frac{m(r^2\Omega)'}{r^2}\right)-\frac{\tilde{\omega}}{\rho}\frac{iD_0}{A}=0. \tag{D5}$$

(D5) does not bear the singularities associated with the Lindblad resonances.

2. Corotation resonance : By definition, $\tilde{\omega}=0$. In order for (A1)~(A5) to have nontrivial solutions, we impose additional conditions

$$\frac{\Pi}{\tilde{\omega}}=\xi,\quad \frac{\omega'}{\tilde{\omega}}=\eta, \tag{D6}$$

where ξ and η are arbitrary finite functions of r. Then, we have

$$\xi D_1=\eta iC, \tag{D7}$$

$$D_1+\rho\eta A=0, \tag{D8}$$

$$2\Omega B=\frac{1}{\rho}C', \tag{D9}$$

$$\frac{(r^2\Omega)'}{r}A+\frac{m}{r\rho}iC=0, \tag{D10}$$

$$\rho A'+\frac{(r\rho)'}{r}A+\frac{m}{r}\rho iB+D_1=0. \tag{D11}$$

There are five equations for four unknown amplitudes. The equation corresponding to (D5) is

$$\left(1-\frac{(r^2\Omega)'}{2r\Omega}\right)\frac{A'}{A}+\frac{(r\rho)'}{r\rho}-\frac{\left(\rho(r^2\Omega)'\right)'}{2\rho r\Omega}=\eta. \tag{D12}$$

η puts the boundary condition. ξ is used to keep the consistency.

At present, whether the dynamical systems (D1)~(D4) and (D7)~(D11) in fact have physically meaningful solutions is an open question.

Appendix E : Taking account of gravity within perturbed disk

The density perturbation gives rise to the modulation of gravity, which in turn modifies the perturbed Euler equations and the continuity equation as follows :

$$\begin{aligned}-i\tilde{\omega}A-2\Omega(B_0+tB_1)+\frac{1}{\rho}\left(C_0'+tC_1'-i\omega' t(C_0+tC_1)\right)-\frac{1}{\rho}(r\Omega^2+f_{Gr})(D_0+tD_1)\\ =\frac{Gr}{\sqrt{2\pi}}\left(D_0'+tD_1'-i\omega' t(D_0+tD_1)\right),\end{aligned} \tag{E1}$$

$$-i\tilde{\omega}B+\frac{(r^2\Omega)'}{r}A+\frac{im}{r\rho}(C_0+tC_1)=im\frac{Gr}{\sqrt{2\pi}\,r}(D_0+tD_1), \tag{E2}$$

$$D_1-i\tilde{\omega}(D_0+tD_1)+\rho A'-\left(i\omega' t\rho-\frac{(r\rho)'}{r}\right)A+\frac{im}{r}\rho B=0, \tag{E3}$$

where the amplitudes C and D have been assumed to be linear in t. Comparing the terms of t^2, t^1 and t^0, we have

t^2 :

$$C_1 = \frac{Gr\rho}{\sqrt{2\pi}} D_1 \tag{E4}$$

t^1 :

$$\frac{1}{\rho}(C_1' - i\omega' C_0) + \frac{\Pi}{\rho} D_1 = \frac{Gr}{\sqrt{2\pi}}(D_1' - i\omega' D_0), \tag{E5}$$

$$\frac{im}{r\rho} C_1 = \frac{imG}{\sqrt{2\pi}} D_1, \tag{E6}$$

$$-i\tilde{\omega} D_1 - \omega' \rho A = 0, \tag{E7}$$

t^0 :

$$-i\tilde{\omega} A - 2\Omega B + \frac{1}{\rho} C_0' + \frac{\Pi}{\rho} D_0 = \frac{Gr}{\sqrt{2\pi}} D'_0, \tag{E8}$$

$$-i\tilde{\omega} B + \frac{(r^2\Omega)'}{r} A + \frac{im}{r\rho} C_0 = im \frac{G}{\sqrt{2\pi}} D_0, \tag{E9}$$

$$D_1 - i\tilde{\omega} D_0 + \rho A' + \frac{(r\rho)'}{r} A + \frac{im}{r} \rho B = 0, \tag{E10}$$

where $\Pi = -r\Omega^2 - f_{Gr}$. Since (E4) and (E6) are equivalent, we are left with six equations for seven unknown functions A, B, C_0, C_1, D_0, D_1 and ω.

We aim to obtain an equation of a single amplitude, say, A. From (E4) and (E7), we readily have

$$D_1 = -\frac{\omega' \rho}{\tilde{\omega}} A, \tag{E11}$$

$$C_1 = -\frac{Gr\rho}{\sqrt{2\pi}} \frac{\omega' \rho}{\tilde{\omega}} A. \tag{E12}$$

Rewrite (E8), (E9) and E(10) as

$$i\tilde{\omega} A + 2\Omega B = \frac{1}{\rho}\left(C_0 - \frac{Gr\rho}{\sqrt{2\pi}} D_0\right)' + \frac{1}{\rho}\left[\left(\frac{Gr\rho}{\sqrt{2\pi}}\right)' + \Pi\right] D_0, \tag{E13}$$

$$\frac{(r^2\Omega)'}{r} A - i\tilde{\omega} B = -\frac{im}{r\rho}\left(C_0 - \frac{Gr\rho}{\sqrt{2\pi}} D_0\right), \tag{E14}$$

$$A' + \left(\frac{(r\rho)'}{r\rho} - \frac{\omega'}{\tilde{\omega}}\right) A + \frac{im}{r} B - i\frac{\tilde{\omega}}{\rho} D_0 = 0, \tag{E15}$$

respectively.

Eliminate C_1 from (E4) and (E5) as

$$\left(\frac{Gr\rho}{\sqrt{2\pi}}\right)' D_1 - i\omega' C_0 + \Pi D_1 = -\omega' \frac{Gr\rho}{\sqrt{2\pi}} D_0$$

Rearranging the terms with a use of (E11), we have

$$C_0 - \frac{Gr\rho}{\sqrt{2\pi}} D_0 = i\left[\left(\frac{Gr\rho}{\sqrt{2\pi}}\right)' + \Pi\right] \frac{\rho}{\tilde{\omega}} A. \tag{E16}$$

Substituting (E16) to (E13) and (E14), with $\varepsilon \equiv \left(Gr\rho/\sqrt{2\pi}\right)'$

$$\tilde{\omega}A - 2\Omega iB + \frac{i}{\rho}(\Pi+\varepsilon)D_0 = \frac{1}{\rho}\left[\left((\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}\right)'A + (\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}A'\right], \tag{E17}$$

$$-i\tilde{\omega}B + \frac{(r^2\Omega)'}{r}A = \frac{m}{r\tilde{\omega}}(\Pi+\varepsilon)A. \tag{E18}$$

(E17), (E18) and (E15) for A, B and D_0 are remaining. (E18) is used to eliminate B from (E15) and (E17) as

$$A' + \left(\frac{(r\rho)'}{r\rho} - \frac{\omega'}{\tilde{\omega}}\right)A + \frac{m}{r\tilde{\omega}}\left[\frac{(r^2\Omega)'}{r} - \frac{m}{r\tilde{\omega}}(\Pi+\varepsilon)\right]A = i\frac{\tilde{\omega}}{\rho}D_0, \tag{E19}$$

$$\rho\tilde{\omega}A - 2\Omega\frac{\rho}{\tilde{\omega}}\left[\frac{(r^2\Omega)'}{r} - \frac{m}{r\tilde{\omega}}(\Pi+\varepsilon)\right]A + (\Pi+\varepsilon)iD_0 = (\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}A' + \left((\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}\right)'A. \tag{E20}$$

respectively.　From (E19) and (E20), one can eliminate $A' - i(\tilde{\omega}/\rho)D_0$ to obtain

$$\begin{aligned}&\left[\rho\tilde{\omega} - 2\Omega\frac{\rho}{\tilde{\omega}}\left(\frac{(r^2\Omega)'}{r} - \frac{m}{r\tilde{\omega}}(\Pi+\varepsilon)\right) - \left((\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}\right)'\right]A \\ &\qquad = -(\Pi+\varepsilon)\frac{\rho}{\tilde{\omega}}\left[\frac{(r\rho)'}{r\rho} - \frac{\tilde{\omega}'}{\tilde{\omega}} + \frac{m}{r\omega'}\left(2\Omega - \frac{m}{r\tilde{\omega}}(\Pi+\varepsilon)\right)\right]A,\end{aligned}$$

where a use has been made of the relation $\omega = \tilde{\omega} + m\Omega$.　Assuming that $A \neq 0$, the eigenfrequency equation (4.10) that takes the self-gravity into account is obtained.

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誤り訂正

東北学院大学教養学部論集 第 171 号 pp 105-146 高橋光一『単純渦と台風』

温度計算のコードに誤りがあり訂正した。結果，p 133 図 6.1 と p 135 図 6.3 を次の図で置き換える。同時に，文中の「図 6.3」をすべて「図 6.2」と読み替える。その他，図の説明文および本文に変更はない。計算の詳細は改めて報告する予定である。

図 6.1

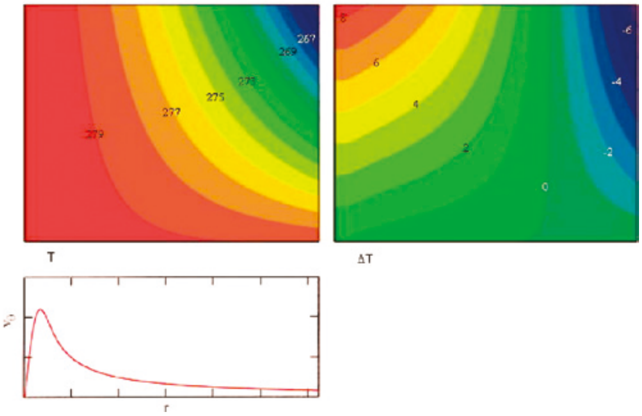

図 6.3（図 6.2 と読み替え）