【Article】

A New Eddy-Viscosity Model and Turbulence

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Abstract : By treating the viscosity in the Navier-Stokes equation as an independent dynamical degree of freedom, a new model for the mean fields in turbulence is proposed. The equation of motion of the viscosity field is determined by resorting to three working hypotheses:i) The viscosity inversion invariance in the Navier-Stokes equation should hold in the new equations. ii) The potential energy of the continuum should be minimized. iii) Both of the action and reaction between the velocity field and the viscosity field should be taken into account. The velocity field and the viscosity field in the model are respectively interpreted as the mean velocity and the eddy-viscosity. When applied to flows in a channel or a circular pipe with large Reynolds numbers, the model semi-quantitatively reproduces the feature of the mean velocity and a Reynolds stress observed in experiments in the whole spatial region without recourse to damping functions introduced by hand in prevalent eddy-viscosity models.

1. Introduction

Fluid bounded by walls shows complex flow patterns called turbulence accompanied with swirlings of various scales that cause fluctuations. Its averaged flow pattern is affected from such small structures as waves and eddies. These are frequently formed through interactions with wall that generates vorticity and effectively modify the viscosity and in turn the property of the mean flow. Owing to such a circumstance, the wall, in addition to the fluid itself, looks in its vicinity like an independent generator of viscosity.

The distinctive viscosity profiles near the wall are conveniently understood in terms of such distinctive physical laws as the Prandtl's wall law and the Kármán's velocity defect law at the vicinity of and the place sufficiently far from the wall, respectively. The logarithmic behaviour of the velocity field can be derived from these laws (Izakson 1937, Milikan 1939), as well as by the Prandtl's mixing-length theory (Prandtl 1933). In phenomenology, it is customary to introduce damping function(s) as was proposed by van Driest (1956) in the expression for the mixing length in order to keep quantitative consistency with the experiments (Laufer 1951, Klebanoff 1954, Wei and Willmarth 1989, Zanoun et al. 2008).

A vast amount of works have been made to understand the flow profile as a whole from not piecewise independent principles but a unified view point. For instance, the k-ε model (Jones and Launder 1972, Launder and Spalding 1974, Bailly and Comte-Bellot 2015) treats the kinetic energy k of the flow and its average dissipation rate ε as independent local fields to be transported, thereby puts the Reynolds equations into a closed and computationally tractable form. Specifically, in the k-ε model, the dissipations of k and ε are governed by a spatially and temporally dependent dissipation coefficient constructed from k and ε themselves. The model is successful in providing good fits to data for mean of turbulence in channel flow in the whole spatial region, at the cost of introducing several adjustable free parameters and functions. In order to gain better fitting to experimental data, models other than the k-ε model have also been proposed, which are mutually discriminated by the physical quantities incorporated in models, the number of parameters and the functional form of damping functions. For later developments in this field, see, e.g., Nagano and Tagawa (1990), Suga (1998) and Karimpour and Venayagamoorthy (2013) and references cited therein. A concise review has been provided by Bredberg (2001).

It is desirable to find succinct and tractable models that enable us to understand the mechanism of turbulence by grasping the fundamental physical processes. Chen et al. (1998) solved their Reynolds-averaged Navier-Stokes (N-S) equation to find analytic expressions of velocity and Reynolds stress in uniform and isotropic turbulence. The N-S equation modified by considering random inelastic molecular collisions as a part of dissipation allows analytic solutions for the mean of turbulence (Jirkovsky and Muriel 2012). There exist other efforts toward understanding the statistical law of turbulence from the first principle. See, e.g., references cited in Frewer et al. (2016). So far, however, some important empirical aspects of turbulence like scaling laws over the larger region of the flow are left unexplained. Apart from studies by computer simulations for engineering purposes, the construction of a model that can be subjected to mathematical analysis with clear physical or mathematical reasoning seems still unsatisfactory.

In this paper, noting that the dissipation coefficient in the k-ε model appears as the effective viscosity, we pursue an alternative possibility of treating the viscosity as an independent dynamical field. This idea, being natural due to the origin of viscosity mentioned at the beginning, also emerges when we notice that changing in the sign of viscosity coefficient still bears a physical meaning. In fact, Lilly (1992) found that the eddy viscosity in the so-called dynamic Smagorinsky model (Germano et al. 1991) the eddy viscosity could be negative and interpreted this phenomena as a result of inverse transport of energy from small to large lattice scale. To this we also add two facts. i) The steady vortex solutions to the N-S equation are allowed after changing the sign of the viscosity (Takahashi

2015). ii) When the sign of the viscosity is changed, the Oseen's vortex (Oseen 1911) strengthens in the course of time, which has many observational counterparts in nature. Since the viscosity field will vary through diffusion and advection, the conventional transport mechanism will play a role to govern the temporal and spatial behaviour of the field. If the viscosity can be effectively treated as an independent field variable, the cumbersome procedure of relating the viscosity to diffusive physical ingredients and other controlling factors in fluid will be greatly lightened.

In the next section, the model for the mean field of turbulence with the dynamical viscosity is presented. In sec.3, the mean velocity for a channel flow is calculated within the model defined in sec.2. In sec.4, one component of the Reynolds stress in a channel flow is calculated and a comparison with the mixing-length theory is made. In sec.5, the result of applying the model to turbulence in a circular pipe is briefly mentioned. The last section is devoted to summary and some remarks.

2. Mean field equation

The N-S equation for the motion of the incompressible fluid with no external body force is given by

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\cdot(\nu\nabla\boldsymbol{u}) - \nabla p/\rho, \tag{2.1}$$

which is constrained by the incompressibility condition

$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.2}$$

$\boldsymbol{u}$ is the velocity field, ρ the density, p the pressure and ν the kinematic viscosity. We replace ν in (2.1) by a space-time dependent function as

$$\nu \rightarrow \bar{\nu}\phi \tag{2.3}$$

with $\bar{\nu}$ being a constant for the representative kinematic viscosity in absence of the velocity gradient. The newly introduced dimensionless function ϕ is a local field that is to express the spatio-temporal variance of viscosity. Then, (2.1) describes how the action of ϕ affects the velocity.

Let us find out the equation that ϕ obeys by resorting to some working hypotheses. When the velocity gradient does not exist, ϕ must be static and unity. Under this circumstance, the equation for ϕ, $Q(\phi)$=0, should give a constant as the solution, say, ϕ=1. For simplicity, we may assume $Q(\phi)$ to be a regular function of ϕ. Next, we require that the invariance of the dynamics under the viscosity inversion $\nu \rightarrow -\nu$ or $\phi \rightarrow -\phi$, which changes dissipation into cohesion, for steady simple vortices be preserved (Takahashi 2015). This means that $Q(\phi)$ is an even function of ϕ. The simplest one that fulfils these requirements is

$$Q(\phi)=\lambda_1(\phi^2-1)/2 \tag{2.4}$$

where λ_1 is a constant.

The deviation of ϕ from unity will be caused by the local fluctuations of the temperature, the pressure and the flow structures (i.e., waves and/or eddies), which are carried away by the flow and diffuse. Together with the advection and the diffusion terms that will describe such processes, the equation for ϕ may be written as

$$\partial_t\phi+\boldsymbol{u}\cdot\nabla\phi=\lambda_0\nabla^2\phi-Q(\phi). \tag{2.5}$$

λ_0 is a positive constant.

However, (2.5) is insufficient in that it does not incorporate the reaction of the velocity field on ϕ that would exist owing to the first term on r.h.s. of (2.1) with (2.3). We shall determine the reaction term by requiring the energy of the steady state be a local minimum. For this purpose, let us note that the N-S equation satisfies the 'variational' equation

$$(\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\,\boldsymbol{u})\cdot\delta\boldsymbol{u}+\delta\boldsymbol{u}\cdot\frac{\delta}{\delta\boldsymbol{u}}\int(\bar{\nu}\phi(\nabla\boldsymbol{u})^2/2+\boldsymbol{u}\cdot\nabla p+\rho)d\boldsymbol{r}=0$$

for small variation $\delta\boldsymbol{u}$. Here, $(\nabla\boldsymbol{u})^2\equiv\sum_{i,j}(\partial_i u_j)^2$. Similarly, for ϕ, (2.5) means the following 'variational' equation

$$(\partial_t\phi+\boldsymbol{u}\cdot\nabla\phi)\delta\phi+\delta\phi\frac{\delta}{\delta\phi}\int[\lambda_0(\nabla\phi)^2/2+\lambda_1(\phi^3/3-\phi)/2]d\boldsymbol{r}=0.$$

Thus, if a unified variational principle existed, it would be equivalent to

$$\delta L\equiv(\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\delta\boldsymbol{u}+\varpi\left(\dot{\phi}+\boldsymbol{u}\cdot\nabla\phi\right)\delta\phi+\delta\int U d\boldsymbol{r},$$

where

$$U(\boldsymbol{u},\ \phi)=\frac{\bar{\nu}}{2}\phi(\nabla\boldsymbol{u})^2+\frac{1}{\rho}\boldsymbol{u}\cdot\nabla p+\varpi\left[\frac{\lambda_0}{2}(\nabla\phi)^2+\frac{\lambda_1}{2}\left(\frac{\phi^3}{3}-\phi\right)\right], \tag{2.6}$$

is 'the specific potential energy' of the continuum. ϖ is a constant with the dimension of velocity squared, which is introduced to adjust the dimension. $\int\delta L dt$ corresponds to the 'variation of the action'. Then, δL=0 under the 'variation' of $\boldsymbol{u}$ yields the expression on r.h.s. of (2.1) from the first and second terms in (2.6).

Similarly, the r.h.s. of (2.5) emerges from the last two terms in (2.6) by the variation of ϕ. In addition to these, the first term on r.h.s. of (2.6) gives rise to a new term $-\bar{\nu}(\nabla\boldsymbol{u})^2/2$. Then, by the same procedure that led to (2.1), we have, instead of (2.5), the equation

$$\partial_t\phi+\boldsymbol{u}\cdot\nabla\phi=\lambda_0\nabla^2\phi-\frac{\lambda_1}{2}(\phi^2-1)-\frac{\overline{\nu}}{2\varpi}(\nabla\boldsymbol{u})^2. \tag{2.7}$$

Because of (2.4), the steady and uniform solutions $\phi = +1$ is locally stable (unstable) if λ_1 is positive (negative) and vice versa for the other solution $\phi = -1$. We assume $\lambda_1 > 0$. Our model is defined by the dynamical system (2.1), (2.2), (2.3) and (2.7). It will later turn out that ϕ is directly related to the eddy viscosity.

In case the system is diffusive, the deterministic least action principle known so far is for Lagrange description in which the path of the fluid element is treated as stochastic variable (Yasue 1981, Nakagomi et al. 1981, Cipriano and Cruzeiro 2007, Constantin and Iyer 2008, Eyink 2010 and references cited therein). In contrast, our procedure that leads to (2.7) is not founded on the standard action principle because δL is not the total derivative. This problem is expected not to cause a serious defect of our model construction, as will be remarked in the last section.

For brevity, we shall call the model defined by (2.1), (2.2), (2.3) and (2.7) the dynamical eddy viscosity model (DEVM). This model is seemingly analogous to Spalart and Allmaras's one equation eddy viscosity model (OEEVM) (Spalart and Allmaras 1992) in that the viscosity is treated as dynamical variable in both approaches. OEEVM, like our model, also includes the quadratic 'destruction' term in the transport equation of the viscosity to take the wall-blocking effect into account. The major difference lies in that, while the common practice in the eddy viscosity models is to employ the Reynolds-averaged N-S equation, our model utilizes the generalized N-S equation for mean flow in which the Reynolds stress does not *explicitly* appear. Our model is also featured by involving no adjustable damping functions, which is in contrast to the prevalent eddy viscosity models including OEEVM. In the eddy viscosity model, the damping functions are introduced by hand to suppress the eddy viscosity in close vicinity of the wall in order to achieve consistency with observations. Then, how is the suppression of the near-wall viscosity expected to take place in our DEVM, if any ?

In uniform channel flow, the last term on r.h.s. of (2.7) is proportional to the square of the velocity gradient. Since ϕ is likely to be limited as $\phi^2 < 1$, a large velocity gradient at the vicinity of the wall will result in a large second derivative of ϕ provided that λ_0 is positive. In other words, when approached toward the wall, ϕ will rapidly decrease and give rise to the near-wall decrease in the eddy viscosity. In other words, in the DEVM, the near-wall suppression of the eddy viscosity is expected to occur as a direct result of the reaction from the velocity gradient.

(2.7) states that the viscosity behaves effectively as a dynamical variable. The spatial and temporal change of the viscosity will be brought about by circumstantial condition that is posed on the fluid. The parameters λ_0, λ_1 and/or ϖ therefore must be related to the experimental condition as well as the physical property of the fluid. The forms of the relations are guessed to some extent by requiring the scaling property of the standard N-S equation to hold in DEVM, too. Namely, if the length and the velocity are measured in units of the characteristic length ℓ and the characteristic velocity V of the flow (time is measured by ℓ/V), then the equation involves these scales via. the Reynolds number $Re = V\ell/\nu$. (2.1) and (2.2) of course fulfil this requirement. As regards (2.7), after the rescaling, the three terms on r.h.s. are accompanied with $(\lambda_0/\nu)Re^{-1}$, $\lambda_1\ell/V$ and $(V^2/\varpi)(\bar{\nu}/\nu)Re^{-1}$, respectively. Thus,

$$\lambda_1 \propto V/\ell, \quad \varpi \propto V^2. \tag{2.8}$$

It may be instructive to compare (2.7) with an analogous transport equation in, e.g., the k-ε model. There, the equation for the dissipation rate ε is expressed as

$$\partial_t\varepsilon + \boldsymbol{u}\cdot\nabla\varepsilon = \nabla\cdot\left(\left(\frac{C_\mu k^2}{\sigma_\varepsilon\varepsilon} + \nu\right)\nabla\varepsilon\right) - C_{\varepsilon 2} f_2 \frac{\varepsilon^2}{k} + \frac{1}{2}C_{\varepsilon 1}C_\mu k\bar{e}_{ik}{}^2, \tag{2.9}$$

where $\boldsymbol{u}$ is the mean velocity, $\bar{e}$ twice the rate-of-strain tensor and f_2 the damping function appropriately chosen for numerical tuning. Other quantities are constant. Comparing (2.7) with (2.9), we readily notice, for constant k, the correspondences

$$-\bar{\nu}(\nabla u)^2/2 \leftrightarrow k\bar{e}_{ik}{}^2 \quad \text{or} \quad -(k^2/\varepsilon^2)(\nabla\varepsilon)^2, \tag{2.10a}$$

$$\phi \leftrightarrow \varepsilon. \tag{2.10b}$$

Thus, (2.7) formally incorporates the turbulent energy k and the energy dissipation rate ε through the gradient of the mean velocity $\boldsymbol{u}$ and the viscosity field ϕ in our dynamical system. This corresponds to the role of the Boussinesq's assumption (Boussinesq 1877) in the eddy viscosity models : the gradient of the mean velocity induces the turbulent stress. This point will be clarified in sec.4 where the near-wall properties are elaborated.

It should be noticed that, since ϕ has space-time dependences, the velocity field $\boldsymbol{u}$ in our equation system is *not* identical to the one appearing in the standard N-S equation (2.1) with a constant ν. Specifically, the solutions to our system, even if they are stationary, generally involve the contributions from gradient-finite ϕ and can not be the laminar flows described by the standard N-S equations. On the basis of the reasoning stated above, we interpret such $\boldsymbol{u}$ as the mean velocity in turbulent flow. This interpretation will be corroborated in what follows by applying the model to the flow with boundaries.

3. Mean viscosity and mean velocity in channel flow

Let us consider a steady parallel flow $\boldsymbol{u} = (u_x(z), 0, 0)$ bounded by two planes $z = 0$ and $2d$ in the Cartesian coordinate. The system is assumed to be uniform in the x- and y-directions. From the equations (2.1) and (2.7), we have

$$(\phi \hat{u}_x')' + \alpha = 0, \quad \alpha \equiv \frac{c_0}{\bar{\nu}} \sqrt{\frac{\lambda_0}{\lambda_1^3}}, \tag{3.1}$$

$$\phi'' - \frac{1}{2}(\phi^2 - 1) - \frac{\beta}{2}(\hat{u}_x')^2 = 0, \quad \beta \equiv \frac{\bar{\nu}\lambda_1}{\varpi}, \tag{3.2}$$

where the dimensionless velocity $\hat{u}_x \equiv (\lambda_0 \lambda_1)^{-1/2} u_x$ has been utilized and the prime stands for a derivative with respect to the dimensionless coordinate $\hat{z} \equiv (\lambda_0/\lambda_1)^{-1/2} z$. It has been assumed that $c_0 \equiv -(dp/dx)/\rho$ is constant. The continuity is automatically satisfied. (3.1) is integrated once to yield

$$\phi \hat{u}'_x = C_1 - \alpha \hat{z}, \tag{3.3}$$

with C_1 being an integration constant. Since $\hat{u}_x' = 0$ at the middle point, C_1 is related to the half channel width d and the parameter α by

$$C_1 = \alpha \hat{d} \equiv \alpha (\lambda_0/\lambda_1)^{-1/2} d. \tag{3.4}$$

Expand $\hat{u}_x$ and ϕ around $\hat{z} = 0$ as

$$\hat{u}_x = \sum_{n=0} \frac{u_n}{n!} \hat{z}^n, \quad \phi = \sum_{n=0} \frac{\phi_n}{n!} \hat{z}^n. \tag{3.5}$$

No-slip implies $u_0 = 0$. Substitute these to (3.1) and (3.2) to obtain the following relations among expansion coefficients that are utilized for setting the boundary conditions at $\hat{z} = 0$:

$$\phi_0 u_1 = C_1, \quad \phi_0 u_2 + \phi_1 u_1 = -\alpha, \quad 2\phi_2 + \phi_0^2 - 1 = \beta u_1^2. \tag{3.6}$$

u_x at the viscous sublayer is usually expressed as $u_x = (u_\tau / l_\tau) z$ or

$$\hat{u}_x = (\hat{u}_\tau / \hat{l}_\tau) \hat{z} = u_1 \hat{z}. \tag{3.7}$$

u_τ is the wall-friction velocity, and l_τ the wall-friction length. Let us introduce a constant γ by

$$\hat{l}_\tau \equiv (\lambda_0/\lambda_1)^{-1/2} l_\tau = \gamma \phi_0. \tag{3.8}$$

Then, comparing (3.7) with (3.5) and (3.6), we also have

$$\hat{u}_\tau \equiv (\lambda_0 \lambda_1)^{-1/2} u_\tau = \hat{l}_\tau C_1/\phi_0 = \gamma C_1. \tag{3.9}$$

From (3.8) and (3.9), if $\gamma \sim O(1)$, $(\lambda_0/\lambda_1)^{1/2}\phi_0$ and $(\lambda_0/\lambda_1)^{1/2}C_1$ provide the measures for l_τ and u_τ, respectively.

Numerically integrating (3.1) and (3.2) with the condition (3.6) is easy and the results are shown in Fig. 1 for $\alpha = 0.008$, 0.01, 0.0126 and 0.014. For other parameters, see the figure caption. We have chosen the values of parameters so as for the equalities $\phi' = \hat{u}_x' = 0$ to hold at the same position in $\hat{z}$.

The values of ϕ as functions of z/d shown in Fig. 1 exhibit a following feature. After a negligible decrease which is not revealed in the figure, ϕ monotonically grows toward the centre of the channel. The growths are approximately linear near the wall. As α gets larger, ϕ in the central region exhibits larger deviation from unity.

By employing the logarithmic scale for $\hat{z}$, the calculated u_x in the vicinity of the wall are shown in the right panel of Fig. 1. The overall feature observed in experiments has been reproduced qualitatively (Laufer 1951, Wei and Willmarth 1989, Zanoun et al. 2003). In particular, the distinction between the regions in which u_x grows linearly ($z/l_\tau < 2$) and almost logarithmically ($z/l_\tau > 50$), which is characterized by the bending of curve in between, is clearly observed. A close look at this figure also reveals a tendency that a larger α yields a larger bending of the curve in the transition layer. The negativity of ϕ_1 in (3.5) seems quite effective to achieve this feature of the velocity distribu-

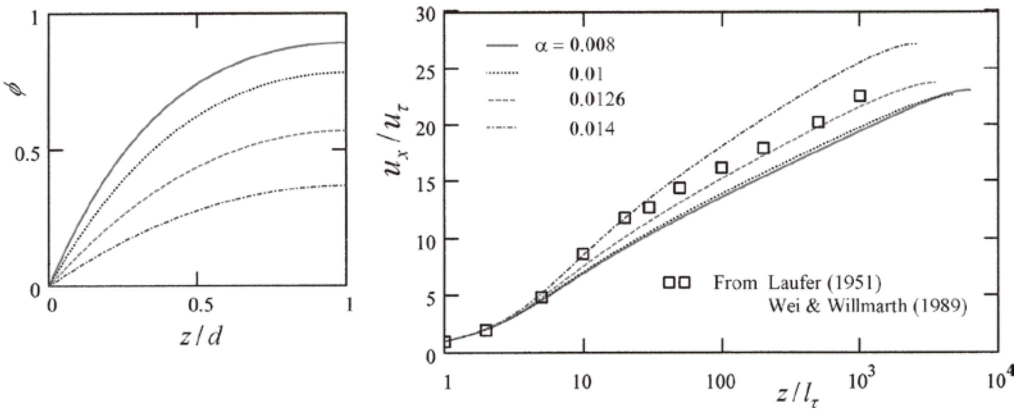

Fig. 1. Solutions to (3.2) and (3.4) with $\lambda_1 > 0$ in the region $0 < z \leq d$. β, ϕ_0 and ϕ_1 are fixed to 1, 0.001 and −0.18, respectively. Left panel : ϕ vs. z/d. Right panel : u_x/u_τ vs. z/l_τ. Solid curve : $(\hat{u}_{x,\max}, \hat{d}, C_1, \hat{u}_\tau)$=(0.29, 3.13, 0.025, 0.013) ; dotted curve : (0.28, 2.44, 0.0244, 0.012) ; dashed curve : (0.26, 1.74, 0.022, 0.011) ; dash-dotted curve : (0.24, 1.28, 0.018, 0.009).

tion. Note that no adjustable functions have been utilized to obtain this result.

The calculated velocity in the logarithmic layer shown in Fig. 1 for smaller α behaves as

$$\frac{\hat{u}_x}{\hat{u}_\tau} \approx \frac{1}{\kappa}\ln(\hat{z}/\hat{l}_\tau)+C=\frac{1}{0.41}\ln(\hat{z}/\hat{l}_\tau)+2.4 \tag{3.10}$$

We see that the Kármán constant $\kappa \approx 0.41$ and $C \approx 2.4$. These are to be compared with the experimentally obtained values $\kappa_{\exp} = 0.37 \sim 0.4$ and $C_{\exp} = 3.7 \sim 5$ (Zanoun et al. 2003, Dean 1978).

4. Reynolds stress

The Reynolds stress characterizes the turbulence by quantifying the deviation from the mean and is governed by the Reynolds equations derived from the standard N-S equation. For the present problem, it is written as

$$\boldsymbol{u}\cdot\nabla u_x + \overline{\delta\boldsymbol{u}\cdot\nabla\delta u_x} = \nu\nabla^2 u_x - \partial_x p/\rho, \tag{4.1}$$

where the bar stands for the ensemble average and δ the deviation from the mean value denoted by unbarred field variables. ν is the constant kinematic viscosity. For the uniform turbulence in incompressible fluid, (4.1) leads to a relation of the Reynolds stress and the mean flow around the centre of the channel (Jones and Launder 1972).

$$-\frac{\overline{\delta\hat{u}_z\delta\hat{u}_x}}{\hat{u}_\tau^2} \approx 1-\frac{z}{d}. \tag{4.2}$$

We assume that the means in (4.1) and the corresponding ones in (2.1) are the same. By subtracting (2.1) with (2.3) from (4.1), the expression for the Reynolds stress in the DEVM is obtained

$$-\overline{\delta\boldsymbol{u}\cdot\nabla\delta u_x} = \nabla\cdot(\bar{\nu}\phi-\nu)\nabla u_x. \tag{4.3}$$

Then, integrating (4.3) once and utilizing (3.9), we have

$$-\frac{\overline{\delta\hat{u}_z\delta\hat{u}_x}}{\hat{u}_\tau^2} = \frac{\nu}{\gamma^2\lambda_0\phi_0}\frac{\phi-\phi_0}{C_1^2}\hat{u}_x', \tag{4.4}$$

where, requiring (4.4) to vanish on the wall, we have set

$$\bar{\nu} = \nu/\phi_0. \tag{4.5}$$

In the central region, (4.4) will behave as (4.2). In Fig. 2, $(\phi-\phi_0)\hat{u}_x'/C_1^2$ are drawn. As ϕ_0 gets smaller, this function in the central region approaches a straight line with the slope of ~ -11. This feature is entirely consistent with the prediction (4.2) of the scaling theory and the experiments (Klebanoff 1954, Zanoun et al. 2003, Jones and Launder 1972). Since the slope of (4.4) as a function of

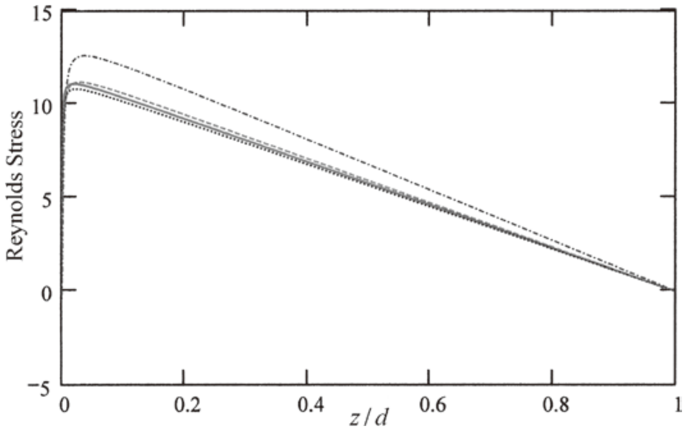

Fig. 2. Plots of $(\phi-\phi_0)\partial_z\hat{u}_x/C_1^2$ vs. z/d calculated from ϕ and $\hat{u}_x$ given in Fig. 1. Meanings of curves are same as in Fig. 1.

z/d is −1 near $z=d$, the result shown in Fig. 3 gives a constraint on the Prandtl number

$$\frac{\nu}{\lambda_0} \approx \frac{\gamma^2\phi_0}{11}. \tag{4.6}$$

The Reynolds number is $Re = 2du_{x,\max}/\nu = (2l_\tau u_\tau/\nu)(d/l_\tau)[u_{x,\max}/u_\tau]$. The first factor on the extreme right is $2\lambda_0\gamma^2\phi_0 C_1/\nu \approx 22C_1$ by using (3.8), (3.9) and (4.6). Together with the calculated values of C_1, d/l_τ and $u_{x,\max}/u_\tau$, we have $\mathrm{Re} \approx 22\cdot 0.025\cdot(3.1/(5\times10^{-4}))\,(0.29/0.013) \approx 8000$ for $\alpha = 0.008$.

According to the mixing-length theory (Prandtl 1933), the Reynolds stress near the wall is given by

$$-\overline{\delta u_z \delta u_x} = l^2(\partial_z u_x)^2 \equiv \nu_t\, \partial_z u_x, \tag{4.7}$$

where l is the mixing length and $\nu_t \equiv l^2\partial_z u_x$ is the eddy viscosity. Comparing (4.7) with (4.4) and (3.9), we have

$$l^2(\partial_z u_x)^2 = (\nu/\phi_0)(\phi-\phi_0)\,\partial_z u_x. \tag{4.8}$$

From (4.7) and (4.8), we have the following expression for the mixing length

$$l = \left(\frac{\nu}{\lambda_1\phi_0}\frac{\phi-\phi_0}{\hat{u}_x'}\right)^{1/2}. \tag{4.9}$$

In case $\phi_1 = 0$, using the fact $\beta u_1 \gg 1$ and eliminating u_1 and ϕ_2 from (3.6) and (3.7), (4.9) gives the linear function for l: $l \approx (\nu\beta C_1/\lambda_0)^{1/2} z/(2\phi_0)$ near the wall as is assumed in the mixing-length theory (Prandtl 1933). If ϕ_1 is negative, we may choose ν_0 in (4.3) to be the minimum of $\bar{\nu}\phi$ to avoid an imaginary l. However, we have seen that ϕ_2 is positive and very large as compared to $|\phi_1|$ in the scaling region, so that l can be regarded as real and linear in z for all practical purposes.

From (4.7) and (4.8), we also have

$$\nu_t = \nu(\phi/\phi_0 - 1), \tag{4.10}$$

which gives the direct relation of ϕ to the eddy viscosity. Since $\partial_z u_x$ is constant at the vicinity of the wall, (4.7) and (4.10) means that the Reynolds stress is the origin of the eddy viscosity and in turn ϕ. Therefore, we can say that the DEVM also describes the relation between the mean velocity gradient and the Reynolds stress.

5. Pipe flow

The flows $\boldsymbol{u} = (0, 0, u_z(r))$ in a circular pipe of the radius R with the central axis at $r = 0$ are similarly obtained in the cylindrical coordinate (r, θ, z). The equations corresponding to (3.1) and (3.2) for channel flow now read

$$\phi \frac{1}{\hat{r}} \partial_{\hat{r}}(\hat{r}\, \partial_{\hat{r}}) \hat{u}_z + \partial_{\hat{r}} \phi \partial_{\hat{r}} \hat{u}_z + \alpha = 0, \quad \alpha \equiv \frac{c_0}{\bar{\nu}} \sqrt{\frac{\lambda_0}{\lambda_1^3}}, \tag{5.1}$$

$$\frac{1}{\hat{r}} \partial_{\hat{r}}(\hat{r} \partial_{\hat{r}}) \phi - \frac{1}{2}(\phi^2 - 1) - \frac{\beta}{2}(\partial_{\hat{r}} \hat{u}_z)^2 = 0, \tag{5.2}$$

with $\hat{r} \equiv (\lambda_0/\lambda_1)^{-1/2} r$ and $\hat{u}_z \equiv (\lambda_0 \lambda_1)^{-1/2} u_z$. c_0 is now for the pressure gradient along the z-direction. These are solved by subjecting $\hat{u}_z$ and ϕ to the expansions in $\hat{r}$ as in (3.5). Starting from $\hat{r} = 0$, integrations are performed until $\hat{u}_z$ vanishes. The point where $\hat{u}_z$ vanishes gives the pipe radius $\hat{R} \equiv (\lambda_0/\lambda_1)^{-1/2} R$. Very abrupt decrease of $\hat{u}_z$ at $\hat{r} = \hat{R}$ occurs and our numerical integration were accompanied with the maximum relative errors of the order of 10^{-2} for $\hat{R}$ and $\phi(\hat{R})$.

The results are shown in Fig. 3 for ϕ and u_z/u_τ as functions of the distance from the central axis and the wall, respectively, for five values of $\phi_0 \equiv \phi(r = 0)$. A uniform pressure gradient has been assumed along z-direction. The pressure gradient parameter α is also varied.

ϕ is almost constant in the central region of the pipe and decreases to a very small value at the wall. As in the channel flow, a smaller ϕ gives rise to a larger second derivative of the velocity and in turn more rapid decreases of the velocity particularly in the sublayer as is shown in Fig. 3. Consequently, larger deviation of ϕ_0 from unity brings about larger bending of the curve of u_z in the transition layer. The Kármán constant decreases as α increases. The conformity with the experiments (Laufer 1953, Ferro 2012) is quite well for $\alpha = 0.008$ and $\phi_0 = 0.974$.

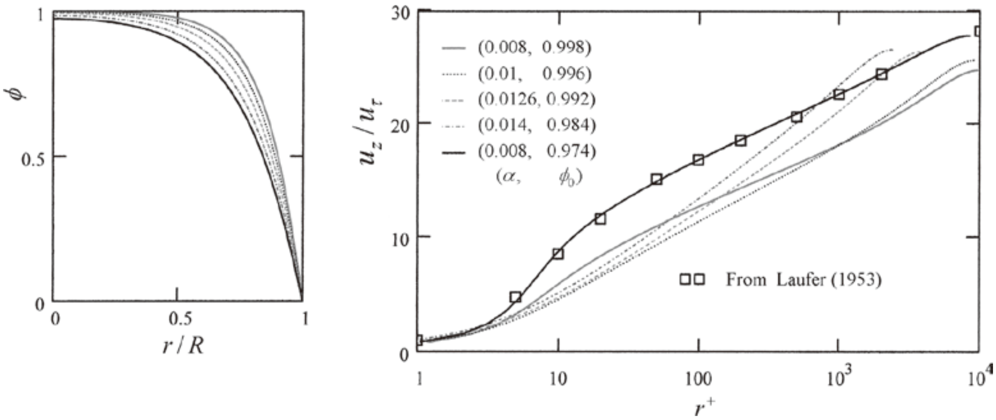

Fig. 3. Flow in a pipe. Left panel : ϕ as a function of r/R where r is the radial distance from the central axis for various α and ϕ_0. The meanings of the curves are designated in the right panel. Right panel : Mean velocity in pipe flow for some parameter values of α and ϕ_0. $r^+ \equiv (R-r)/l_\tau$. $\phi_0 = 0.04$. $(\hat{R}, \hat{u}_{\max}) = (8.6, 0.53)$ for ed solid curve ; (7.8, 0.5) for blue dotted curve ; (7, 0.46) for green dashed curve ; (6.2, 0.014) for purple dot-dashed curve ; (5.7, 0.37) for black solid curve. Squares are taken from Laufer (1953).

6. Summary and remarks

In the DEVM of turbulence, the viscosity field ϕ is an independent degree of freedom. The equation of motion of ϕ was determined by requiring the potential energy be a minimum. The DEVM provides a clear view of action and reaction between the shear and the viscosity. When applied to the steady channel and pipe flows, the results of the model calculations agreed fairly well with the experiments for the mean velocity over the whole spatial range, if the boundary conditions and the model parameters are appropriately chosen. This strongly supports the anticipation that the time-independent velocity in the DEVM is interpreted as the mean velocity of turbulent flow with large Reynolds number.

In addition to the kinematic viscosity, the DEVM involves three model parameters and *no* adjustable functions. The physically interesting flows at the central positions are found around the stable point of the potential of ϕ, but near the wall, largely deviate from the stable point. In this way, ϕ gives rise to the effect of damping factor (Van Dries 1956) that frequently utilized in prevalent eddy viscosity models (Nagano and Tagawa 1990, Suga 1998, Karimpour and Venayagamoorthy 2013, Bredberg 2001).

In the channel flow, the bending of the curves exhibiting the transition from the linear to the logarithmic velocity distribution near the wall is brought about mainly by the negativity of ϕ_1 or the smallness of α (i.e., the smallness of the pressure gradient). The negativity of ϕ_1 means that the viscosity field first decreases with the distance from the wall. At present, what this physically means is not clear. Following the mixing-length theory of turbulence near the wall (Prandtl 1933), the field ϕ is envisaged to express the variation of the viscosity due to generation of eddies with various scales. The negativity and the smallness of ϕ_1 may then mean that in the layer just above the wall the distribution of eddies is stationary. Eddies are accompanied with heat (Fulton 1950, Rott 1959, Mayer and Powell 1992, Bershader 1995, Polihronov and Straatman 2015 2012). However, the generated heat alone is unable to give rise to the viscosity change of order 10^3 over the fluid.

The equations of motion in our DEVM were derived by requiring δL defined in sec.2 to vanish. Unfortunately, δL is not a total derivative, so that this procedure does not constitute the standard least action principle. This is an expected situation because we have adopted the Eulerian description of fluid. One thus might be dubious if the action-reaction relation between the velocity and the viscosity was correctly implemented in the model. Interestingly, it is possible to construct an Eulerian 'action' whose minimization yields equations of motion very similar to those we have discussed in this paper. Our DEVM will be validated in this respect. This feature of our modelling will be reported in a separate paper.

The DEVM can be said promising as long as the steady mean flow of turbulence is concerned, although wide portion in the parameter space is left unexplored. The variants of the form of interaction between the velocity gradient and the viscosity field together with the form of $Q(\phi)$ are to be explored. The crucial point is that the local Eulerian field theory is in fact possible. In this paper, only steady flows were considered. In order for the solutions to be of the really steady, well-developed turbulence, the mean flow must be stable. Investigating the stability of the solution is important and is left for future study.

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